

AN EXTREMAL PROPERTY OF TRIGONOMETRIC POLYNOMIALS

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ABSTRACT. In this article the solution of the special problem of the conditional extremum for the conjugate trigonometric polynomials is given.

$$\sup_{\substack{(a_1, \dots, a_n) \\ \sum a_j = 1}} \min_t \left\{ \sum_{j=1}^n a_j \cos jt : \sum_{j=1}^n a_j \sin jt = 0 \right\} = -tg^2 \frac{\pi}{2(n+1)}.$$

A possibility to apply this result to the problems of optimal stabilization of quasidynamic chaos in discrete systems is mentioned.

As we know, the problem of optimal impact on chaotic regime is one of the most fundamental in nonlinear dynamics [1]. For the multiparameter families of discrete systems this problem can be reduced to the choice of a direction that provides maximum stability in one-parameter space. When we change this parameter we can explore the sequence of bifurcations that leads to occurrence of a chaotic attractor.

First, bifurcation values of the parameter corresponds to the loss of stable equilibrium position in the system. These values are related to the area of Schur stability of family of polynomials,

$$(1) \quad \left\{ f(\lambda) = \lambda^n + k(a_1 \lambda^{n-1} + \dots + a_n), \quad \sum_{j=1}^n a_j = 1 \right\}$$

where k is a parameter. All polynomials of the family (1) are stable for $k = 0$. Moreover, there exist two positive constants k_1, k_2 which depend on the coefficients a_1, \dots, a_n , such that the family remains stable for $k \in (-k_1, k_2)$ and the stability disturbs if $k = k_2 + \varepsilon$ or $k = -k_1 - \varepsilon$, $\forall \varepsilon > 0$. We need to maximize the length of robust stability segment i.e. the function

$$(2) \quad \Phi(a_1, \dots, a_n) = k_1(a_1, \dots, a_n) + k_2(a_1, \dots, a_n).$$

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Function $\Phi(a_1, \dots, a_n)$ has simple geometrical meaning as

$$\frac{f(e^{it})}{ke^{int}} = \frac{1}{k} + \sum_{j=1}^n a_j \cos jt - i \sum_{j=1}^n a_j \sin jt.$$

The points of intersection of curve

$$\left\{ x = \sum_{j=1}^n a_j \cos jt, y = - \sum_{j=1}^n a_j \sin jt \right\}$$

on the OXY plane with OX axis correspond to those values of parameter k , for which the polynomial of the family (1) has zeros on the unit circle. The length of the longest segment which is defined by these points of intersection is $\frac{1}{k_1} + \frac{1}{k_2}$.

Since $\sum_{j=1}^n a_j = 1$, $f(1) = 1+k$. Hence, $\max_{a_1, \dots, a_n} \{ k_1(a_1, \dots, a_n) \} \leq 1$.

Let

$$(3) \quad I = \sup_{a_1, \dots, a_n} \min_t \left\{ C(t) : S(t) = 0, \sum_{j=1}^n a_j = 1 \right\}$$

We have that

$$\max_{a_1, \dots, a_n} \Phi(a_1, \dots, a_n) \leq 1 - \frac{1}{I}.$$

Note, that the value of I is negative as it follows from the Lemma 1.

In order to find the value of I , we need the results that are included in the following six lemmas.

Let

$$(4) \quad \rho = \min_t \left\{ C(t) : S(t) = 0, \sum_{j=1}^n a_j = 1 \right\}.$$

Since the polynomials $C(t)$ and $S(t)$ are trigonometric, it is enough to consider the minimum in (4) on the segment $[0, \pi]$.

Lemma 1. *The value of ρ is negative.*

Proof. Let function $F(z) = \sum_{j=1}^n a_j z^j$, where z is a complex variable. It is clear that $z_0 = 0$ is a zero of $F(z)$.

Hence, when the point z goes along the unit circle once, the argument of the function increases at least by 2π . This means that the plot of $x + iy = F(e^{it})$ on the plane OXY with $t \in [0, 2\pi]$ surrounds zero. Since the function $F(e^{it})$ is continuous, there exists a $t_0 \in [0, 2\pi]$ such that $\operatorname{Re} \{ F(e^{it_0}) \} = C(t_0) < 0$, $\operatorname{Im} \{ F(e^{it_0}) \} = S(t_0) = 0$.

This completes the proof of Lemma 1. \square

Lemma 2. *Let $S(t_0) = 0$. Then the trigonometric polynomial $S(t)$ is presented uniquely by $S(t) = (\cos t - \cos t_0) \sum_{j=1}^{n-1} a'_j \sin jt$ and $(1 - \cos t_0) \sum_{j=1}^{n-1} a'_j = 1 + \frac{a'_1}{2}$*

Proof. Coefficients a_1, \dots, a_n and a'_1, \dots, a'_{n-1} are connected by relations

$$(5) \quad \begin{cases} a_1 = -\cos t_0 \cdot a'_1 + \frac{1}{2}a'_2, \\ a_2 = \frac{1}{2}a'_1 - \cos t_0 \cdot a'_2 + \frac{1}{2}a'_3, \\ \dots \\ a_{n-1} = \frac{1}{2}a'_{n-2} - \cos t_0 \cdot a'_{n-1}, \\ a_n = \frac{1}{2}a'_{n-1} \end{cases}$$

The determinant of this system modulo first equation is equal to 2^{1-n} . So, the coefficients a'_1, \dots, a'_{n-1} are uniquely presented via the coefficients a_2, \dots, a_n . Thus, from the normalization requirements we obtain:

$$1 = \sum_{j=1}^n a_j = -\cos t_0 \cdot \sum_{j=1}^{n-1} a'_j + \sum_{j=2}^{n-1} a'_j = (1 - \cos t_0) \cdot \sum_{j=1}^{n-1} a'_j - \frac{1}{2}a'_1.$$

This proves Lemma 2. \square

Lemma 3. *Let $S(t_0) = 0$, $t_0 \in (0, \pi)$. Then*

$$C(t) = -\frac{a'_1}{2} + (\cos t - \cos t_0) \sum_{j=1}^{n-1} a'_j \cos jt.$$

So $C(t_0) = -\frac{a'_1}{2}$, where the coefficients a'_1, \dots, a'_{n-1} are determined via the coefficients a_2, \dots, a_n in the system of equations (5).

Proof. It follows from the system (5) that

$$\begin{aligned} C_n(t) &= \sum_{j=1}^n a_j \cos jt = (-\cos t_0 a'_1 + \frac{1}{2}a'_2) \cos t + \dots + (\frac{1}{2}a'_{n-3} - \cos t_0 a'_{n-2}) \cos(n-2)t + (\frac{1}{2}a'_{n-2} - \cos t_0 a'_{n-1}) \cos(n-1)t + \frac{1}{2}a'_{n-1} \cos nt \\ &= C_{n-1}(t) + \frac{1}{2}a'_{n-1}(\cos(n-2)t - 2\cos t_0 \cos(n-1)t + \cos nt) = C_{n-1}(t) + a'_{n-1} \cos(n-1)t(\cos t - \cos t_0) = C_2(t) + (\cos t - \cos t_0) \sum_{j=2}^{n-1} a'_j \cos jt \\ &= -\cos t_0 a'_1 \cos t + \frac{a'_1}{2} \cos 2t + (\cos t - \cos t_0) \sum_{j=2}^{n-1} a'_j \cos jt = -\frac{a'_1}{2} + (\cos t - \cos t_0) \sum_{j=1}^{n-1} a'_j \cos jt. \end{aligned}$$

Hence $C_n(t_0) = -\frac{a'_1}{2}$. \square

Lemma 4. *Let $S(t_0) = 0$, $t_0 \in (0, \pi)$. Then*

$$C(\pi) = -(1 + \cos t_0)(-a'_1 + a'_2 - \dots) - \frac{a'_1}{2}.$$

Lemma 4 is a consequence of Lemma 3.

Lemma 5. *Let $S(t_0) = S(t_1) = 0$, $t_0 \in (0, \pi)$, $t_1 \in (0, \pi)$, $t_0 \neq t_1$. Then $S(t)$ is presented uniquely in the next form:*

$$S(t) = (\cos t - \cos t_0)(\cos t - \cos t_1) \sum_{j=1}^{n-2} a_j'' \sin jt,$$

and

$$(1 - \cos t_0)(1 - \cos t_1) \sum_{j=1}^{n-2} a_j'' - (1 - \cos t_0 - \cos t_1) \frac{a_1''}{2} - \frac{a_2''}{4} = 1.$$

For the proof we need to apply the Lemma 2 twice.

Lemma 6. *Let $S(t_0) = S(t_1) = 0$, $t_0 \in (0, \pi)$, $t_1 \in (0, \pi)$, $t_0 \neq t_1$. Then*

$$\begin{aligned} C(t_0) &= -\frac{a_2''}{4} + \frac{a_1''}{2} \cos t_1, \\ C(t_1) &= -\frac{a_2''}{4} + \frac{a_1''}{2} \cos t_0. \end{aligned}$$

Proof. By Lemma 2

$$S(t) = (\cos t - \cos t_0) \cdot \sum_{j=1}^{n-1} a_j' \sin jt,$$

where $a_1' = \frac{a_2''}{2} - a_1'' \cos t_1$ by Lemma 5. Then by Lemma 3

$$C(t_0) = -\frac{a_1'}{2} = -\frac{a_2''}{4} + \frac{a_1''}{2} \cos t_1.$$

We get the value of $C(t_1)$ in a similarly way. □

Corollary 1. *If $S(t_0) = S(t_1) = 0$, $C(t_0) = C(t_1)$, $t_0 \neq t_1$, then*

$$C(t_0) = C(t_1) = -\frac{a_2''}{4}.$$

Now we come to the main result of the paper:

Theorem 1. *Let $C(t)$, $S(t)$ be the pair of conjugate trigonometric polynomials defined by*

$$C(t) = \sum_{j=1}^n a_j \cos jt, \quad S(t) = \sum_{j=1}^n a_j \sin jt,$$

satisfying the normalization condition $\sum_{j=1}^n a_j = 1$. Denote by I the conditional extremum $\sup_{a_1, \dots, a_n} \min_t \{ C(t) : S(t) = 0 \}$, we have that

$$(6) \quad I = -tg^2 \frac{\pi}{2(n+1)}.$$

Proof. The function $S(t)$ vanishes at $t = \pi$ for all coefficients a_1, \dots, a_n . In the following, we will find the value $\sup \rho(a_1, \dots, a_n)$ on the set

$$A_R = \left\{ (a_1, \dots, a_n) : \sum_{j=1}^n a_j = 1, \sum_{j=1}^n |a_j| \leq R \right\}.$$

The function $\rho(a_1, \dots, a_n)$ is continuous on the set A_R , except those points (a_1, \dots, a_n) at which the minimum $C(t)$ is achieved at the zero of $S(t)$, where the function $S(t)$ does not change the sign.

Note that the lower limit of the function $\rho(a_1, \dots, a_n)$ at the points of discontinuity is the value of the function, this means that on the set A_R the function $\rho(a_1, \dots, a_n)$ is lower semicontinuous.

Together with the function $\rho(a_1, \dots, a_n)$ we consider the function

$$\rho_1(a_1, \dots, a_n) = \min_{t \in [0, \pi]} \left\{ C(t) : t = T \cup \{\pi\} \right\},$$

where T is set of points of interval $(0, \pi)$ at which the function $S(t)$ changes the sign. The set $T \cup \{\pi\}$ is a subset of the set of all zeros of $S(t)$. Therefore, $\bar{\rho} \leq \bar{\rho}_1$, where $\bar{\rho}$ and $\bar{\rho}_1$ are given, respectively, by
$$\bar{\rho} \leq \sup_{(a_1, \dots, a_n) \in A_R} \{ \rho(a_1, \dots, a_n) \}, \quad \bar{\rho}_1 \leq \sup_{(a_1, \dots, a_n) \in A_R} \{ \rho_1(a_1, \dots, a_n) \}.$$

The function $\rho_1(a_1, \dots, a_n)$ is upper semicontinuous, so on the set A_R it reaches its maximum value, i.e. $\bar{\rho}_1 = \max_{(a_1, \dots, a_n) \in A_R} \{ \rho_1(a_1, \dots, a_n) \}.$

A pair of trigonometric polynomials $(C^0(t), S^0(t))$ on which the maximum is reached is called an optimal pair.

Define for the optimal polynomial $S^0(t)$ the set $T = \{ t_0, t_1, \dots, t_q \}$, where $0 \leq q \leq n-2$. Besides, let

$$\min \{ C^0(t_0), \dots, C^0(t_q) \} = C^0(t_0),$$

and $C^0(t_0) < C^0(t_j)$, $j = 1, \dots, q$, $C^0(t_0) < C^0(\pi)$.

Then by Lemma 2

$$S^0(t) = (\cos t - \cos t_0) \sum_{j=1}^{n-1} a'_j \sin jt.$$

Consider the set $T_0 = \{ \theta, t_1, \dots, t_q \}$ and corresponding pair of trigonometric polynomials $(C_\theta(t), S_\theta(t))$, where

$$S_\theta(t) = \frac{\cos t - \cos \theta}{(1 - \cos \theta)\sigma - \frac{a'_1}{2}} \sum_{j=1}^{n-1} a'_j \sin jt$$

with $\sigma = \sum_{j=1}^{n-1} a'_j$. The normalizing factor in the polynomial $S_\theta(t)$ is selected from lemma 2. It is clear that $S_{t_0}(t) \equiv S^0(t)$.

From Lemma 3 it follows that

$$C_\theta(\theta) = -\frac{\frac{a'_1}{2}}{(1 - \cos \theta) \cdot \sigma - \frac{a'_1}{2}}, \quad \theta \in [t_0, \pi).$$

According to Lemma 1, the value $C^0(t_0) = -\frac{a'_1}{2}$ is negative. Hence $a'_1 > 0$ and $\sigma = \frac{1 + \frac{a'_1}{2}}{1 - \cos t_0} > 0$. Under these conditions the value of $C_\theta(\theta)$ increases as θ increases and doesn't exceed $-\frac{a'_1}{4\sigma - a'_1}$. From the continuity of the functions $C(t), S(t)$ it follows that if $0 < \theta - t_0 < \varepsilon$, then $C_\theta(\theta) > C^0(t_0)$, $|C_\theta(t_j) - C^0(t_j)| < \delta$, $j = 1, \dots, q$, and $|C_\theta(\pi), \dots, C^0(\pi)| < \delta$ for any small ε . Then

$$\min \{ C_\theta(\theta), \dots, C_\theta(t_q), C_\theta(\pi) \} > \min \{ C^0(t_0), \dots, C^0(t_q), C^0(\pi) \} = C^0(t_0)$$

at least for sufficiently small positive value of the difference $\theta - t_0$. This means that the pair of polynomials $C^0(t), S^0(t)$ cannot be optimal.

Consider another case:

$$C^0(\pi) \leq C^0(t_0) < C^0(t_j), \quad j = 1, \dots, q.$$

For polynomials $C_\theta(t), S_\theta(t)$ the equality below holds by Lemma 4:

$$C_\theta(\pi) = -\frac{(1 + \cos \theta)\sigma' + \frac{a'_1}{2}}{(1 - \cos \theta) \cdot \sigma - \frac{a'_1}{2}},$$

where $\sigma' = -a'_1 + a'_2 - \dots$. If $\sigma' \leq -\frac{a'_1 \sigma}{4\sigma - a'_1}$, then $C_\theta(\pi) \geq -\frac{a'_1}{4\sigma - a'_1} > C_\theta(\theta)$, $\theta \in (t_0, \pi)$. Hence $\sigma' > -\frac{a'_1 \sigma}{4\sigma - a'_1}$. In this case the values $C_\theta(\pi), C_\theta(\theta)$ increases as θ increases. So the pair $(C^0(t), S^0(t))$ cannot be an optimal one.

Let $C^0(t_0) = C^0(t_1) < C^0(t_j)$, $j = 2, \dots, q, t_0 \neq t_1$. Then as a consequence of Lemma 6, $a''_1 = 0$. Consider pair of polynomials $(C_{\theta_1, \theta_2}(t), S_{\theta_1, \theta_2}(t))$, where

$$S_{\theta_1, \theta_2}(t) = \frac{(\cos t - \cos \theta_1)(\cos t - \cos \theta_2)}{(1 - \cos \theta_1)(1 - \cos \theta_2)\sigma'' - \frac{a''_2}{4}} \sum_{j=2}^{n-2} a''_j \sin jt.$$

Here $\sigma'' = \sum_{j=2}^{n-2} a''_j$. It is clear that $S_{t_0, t_1}(t) = S^0(t)$.

Then by lemma 5

$$C_{\theta_1, \theta_2}(\theta_j) = -\frac{\frac{a''_2}{4}}{(1 - \cos \theta_1)(1 - \cos \theta_2)\sigma'' - \frac{a''_2}{4}}, \quad j = 1, 2.$$

Because $C^0(t_0) < 0$, so $a_2'' > 0$, $\sigma'' > 0$. In this case the functions $C_{\theta_1, \theta_2}(\theta_j)$ increases with respect to θ_1 and θ_2 . Again, the pair $(C^0(t), S^0(t))$ cannot be the optimal one.

We can make an analogical conclusion in the case when there are more than 2 minimal elements in the set $\{C^0(t_0), \dots, C^0(t_q), C^0(\pi)\}$. So it is shown, that the set T is an empty set, i.e. for the optimal pair it is necessarily satisfied $S^0(t) \geq 0$, $t \in [0, \pi]$.

The trigonometric polynomial $S^0(t)$ can be performed in the form

$$S^0(t) = \sin t(\gamma_1 + 2\gamma_2 \cos t + \dots + 2\gamma_n \cos(n-1)t),$$

where $\gamma_1 = a_1 + a_3 + \dots$, $\gamma_2 = a_2 + a_4 + \dots$, $\gamma_3 = a_3 + a_5 + \dots$, $\gamma_4 = a_4 + \dots$.

There is a bijection between a_1, \dots, a_n and $\gamma_1, \dots, \gamma_n$. The normalization condition $\sum_{j=1}^n a_j = 1$ is equivalent to the equality $\gamma_1 + \gamma_2 = 1$.

Because $T = \emptyset$, then

$$\bar{\rho}_1 = \max_{(a_1, \dots, a_n) \in A_R} \{C(\pi)\} = \max_{(a_1, \dots, a_n) \in A_R} \{-a_1 + a_2 - a_3 + \dots\}.$$

Note that $-a_1 + a_2 - a_3 + \dots = -\gamma_1 + \gamma_2$.

It follows from [2] that the polynomial $S^0(t)$ is nonnegative implies that $|\gamma_2| \leq \cos \frac{\pi}{n+1} \cdot |\gamma_1|$. Then

$$\bar{\rho}_1 = \max_{\gamma_1, \gamma_2} \left\{ -\gamma_1 + \gamma_2 : \gamma_1 + \gamma_2 = 1, |\gamma_2| \leq \cos \frac{\pi}{n+1} |\gamma_1| \right\}.$$

The constrained maximum is reached when

$$\gamma_1^0 = \frac{1}{1 + \cos \frac{\pi}{n+1}}, \quad \gamma_2^0 = \frac{\cos \frac{\pi}{n+1}}{1 + \cos \frac{\pi}{n+1}},$$

And equals to

$$\bar{\rho}_1 = -\frac{1 - \cos \frac{\pi}{n+1}}{1 + \cos \frac{\pi}{n+1}} = -tg^2 \frac{\pi}{2(n+1)}.$$

The polynomial $\frac{S^0(t)}{\sin t} = \gamma_1^0 + 2\gamma_2^0 \cos t + \dots + 2\gamma_n^0 \cos(n-1)t$ is called nonnegative Feier polynomial, and its coefficients are determined uniquely. Hence coefficients a_1^0, \dots, a_n^0 are defined uniquely: $a_1^0 = \gamma_1^0 - \gamma_3^0$, $a_2^0 = \gamma_2^0 - \gamma_4^0$, $a_3^0 = \gamma_3^0 - \gamma_5^0 \dots$. And they are positive and doesn't depend on R because $\sum_{j=1}^n |a_j| = 1$.

It means that for all a_1, \dots, a_n with $\sum_{j=1}^n |a_j| = 1$, we have the followig inequalities

$$\rho_1(a_1, \dots, a_n) \leq \bar{\rho}_1, \quad \rho(a_1, \dots, a_n) \leq \bar{\rho} \leq \bar{\rho}_1.$$

We also need to show that for function $\rho(a_1, \dots, a_n)$ the sup is achieved and equals to $\bar{\rho}_1$. Let us consider one-parameter family of trigonometric polynomials

$$S^\varepsilon(t) = \frac{a_1^0 + \varepsilon}{1 + \varepsilon} \sin t + \frac{a_2^0}{1 + \varepsilon} \sin 2t + \dots + \frac{a_n^0}{1 + \varepsilon} \sin nt.$$

It is clear that $\frac{a_1^0 + \varepsilon}{1 + \varepsilon} + \frac{a_2^0}{1 + \varepsilon} + \dots + \frac{a_n^0}{1 + \varepsilon} = 1$ and $S^\varepsilon(t) = \frac{1}{1 + \varepsilon} S^0(t) + \frac{\varepsilon}{1 + \varepsilon} \sin t$. For all $t \in (0, \pi)$ and $\varepsilon > 0$ holds inequality $S^\varepsilon(t) > 0$. Because $C^\varepsilon(\pi) = \frac{1}{1 + \varepsilon} \bar{\rho}_1 - \frac{\varepsilon}{1 + \varepsilon}$, $C^\varepsilon(\pi) < \bar{\rho}_1$ and $C^\varepsilon(\pi) \rightarrow \bar{\rho}_1$ at $\varepsilon \rightarrow 0$. These conditions and the independence of the coefficients of R means that

$$I = \bar{\rho} = \bar{\rho}_1 = \sup_{\substack{a_1, \dots, a_n \\ \sum a_j = 1}} \{ \rho(a_1, \dots, a_n) \} = -tg^2 \frac{\pi}{2(n+1)}.$$

This completes the proof of the theorem. \square

It is possible to find the optimal coefficients a_1^0, \dots, a_n^0 in an explicit way. Indeed, polynomial $\frac{S^0(t)}{\sin t}$ is proportional to the Feier polynomial:

$$\begin{aligned} \frac{S^0(t)}{\sin t} &= \frac{1}{1 + \cos \frac{\pi}{n+1}} + \frac{2 \cos \frac{\pi}{n+1}}{1 + \cos \frac{\pi}{n+1}} \cos t + \dots = \\ &= \frac{1 - \cos \frac{\pi}{n+1}}{n+1} \cdot \frac{2 \cos^2 \frac{n+1}{2} t}{(\cos t - \cos \frac{\pi}{n+1})^2} = \gamma_1^0 + 2\gamma_2^0 \cos t + \dots \end{aligned}$$

Hence the coefficients $\gamma_1^0, \dots, \gamma_n^0$, a_1^0, \dots, a_n^0 are defined by formulas

$$\begin{aligned} \gamma_j^0 &= \frac{1}{2(n+1) \sin \frac{\pi}{n+1} (1 + \cos \frac{\pi}{n+1})} \left((n-j+3) \sin \frac{\pi j}{n+1} - (n-j+1) \sin \frac{\pi(j-2)}{n+1} \right), \\ a_j^0 &= 2 \cdot tg \frac{\pi}{2(n+1)} \cdot \left(1 - \frac{j}{n+1} \right) \cdot \sin \frac{\pi j}{n+1}, \quad j = 1, \dots, n. \end{aligned}$$

Returning to the problem of maximizing the function (2), we get

$$\max_{a_1, \dots, a_n} \{ k_2(a_1, \dots, a_n) \} = k_2(a_1^0, \dots, a_n^0) = ctg^2 \frac{\pi}{2(n+1)}.$$

Note that coefficients a_1^0, \dots, a_n^0 are positive, so

$$\max_{a_1, \dots, a_n} \{ k_1(a_1, \dots, a_n) \} = k_1(a_1^0, \dots, a_n^0) = 1.$$

Finally,

$$\max_{\substack{(a_1, \dots, a_n) \\ \sum a_j = 1}} \Phi(a_1, \dots, a_n) = 1 + ctg^2 \frac{\pi}{2(n+1)} = \frac{1}{\sin^2 \frac{\pi}{2(n+1)}}.$$

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